Paper:

Physics-Informed Deep B-Spline Networks

Code:

Zhuoyuan Wang, Raffaele Romagnoli, Saviz Mowlavi, Yorie Nakahira

Carnegie Mellon University | Duquesne University | Mitsubishi Electric Research Laboratories

Motivation

Solving **parametric** partial differential equations (**PDEs**) with **varying** initial and boundary conditions (**ICBCs**) is challenging.

- Classical solvers need to rerun for each parametric configuration.
- Physics-informed learning solvers often struggle with enforcing ICBCs with soft loss penalties.
- Hard ICBC constraint (ansatz) approaches can restrict model flexibility and potentially degrade interior solution accuracy.

Method

Parametric PDE formulation:

- Ground truth PDE solution s with state-time $x \in \mathbb{R}^d$, satisfying the following physics law and ICBC

$$\mathcal{F}(s, x, u) = 0, x \in \Omega(\alpha)$$

$$\mathcal{B}(s, x, u) = 0, x \in \partial\Omega(\alpha)$$

where u is the PDE parameter and α is the ICBC parameter.

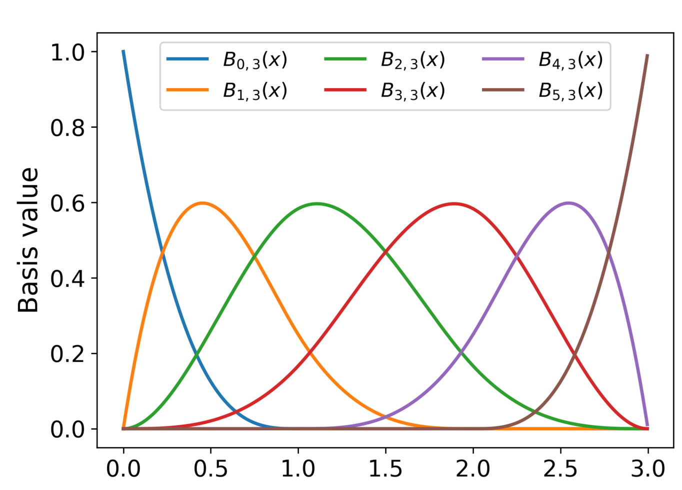
Key insight: leverage **B-spline representation** in **physics-informed learning**

B-spline basis functions

generated through Cox-de Boor formula

$B_{i,d}(x)$: i-th basis with order d

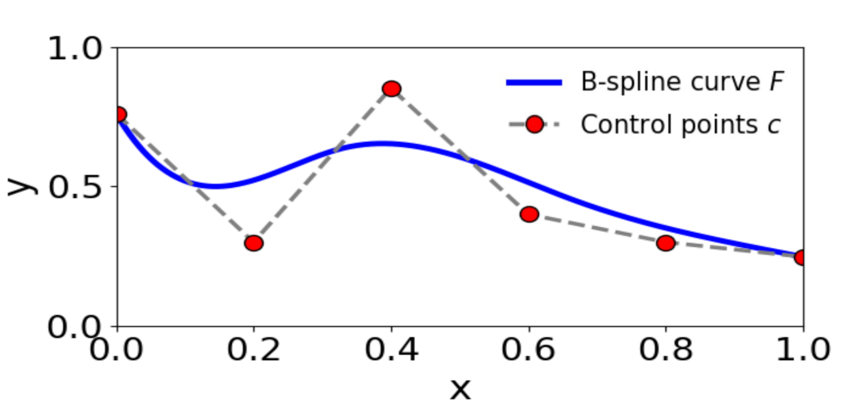
c: weights of the B-spline (control points)



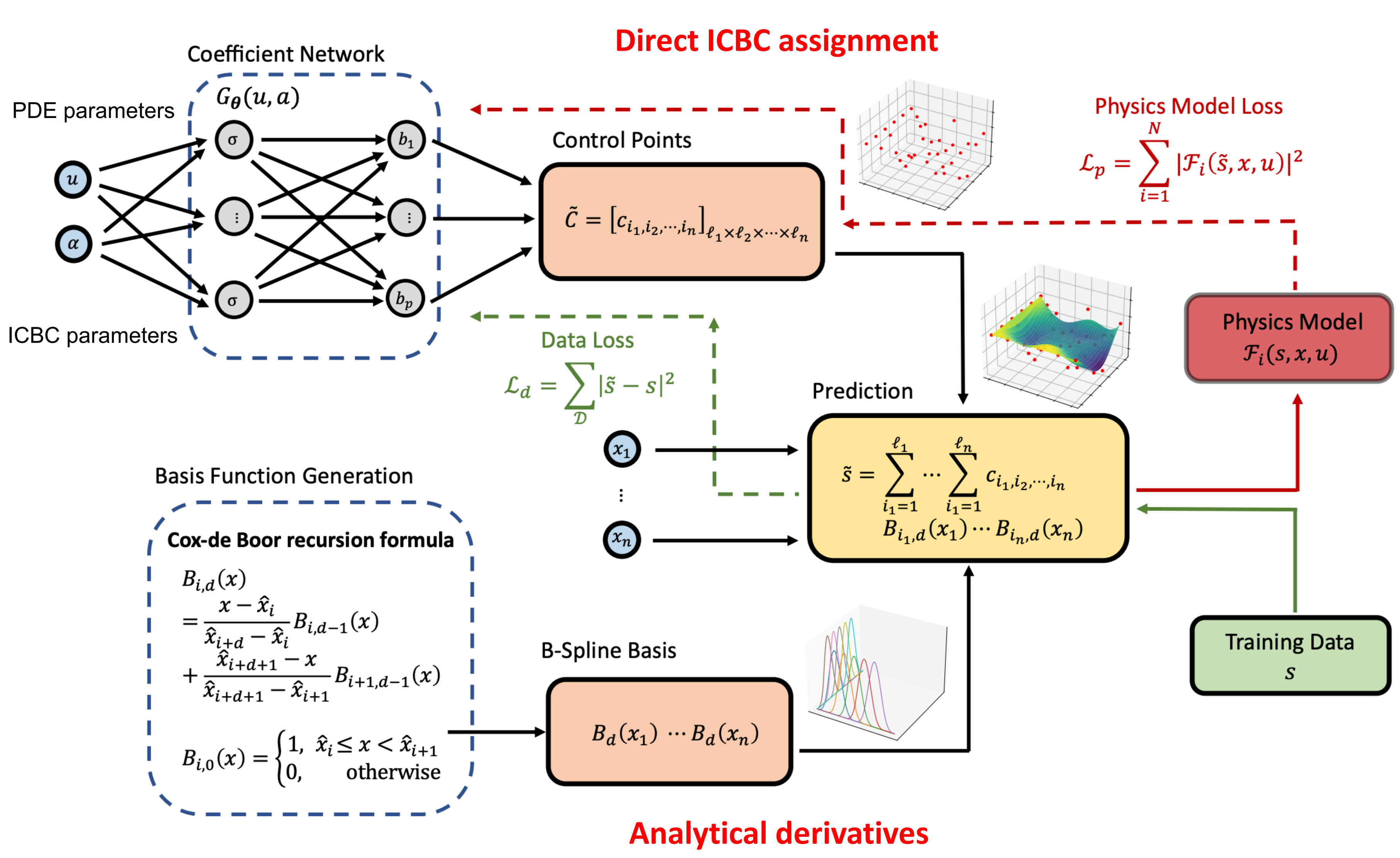
B-spline approximation w/ boundary compliance

with clamped knot points

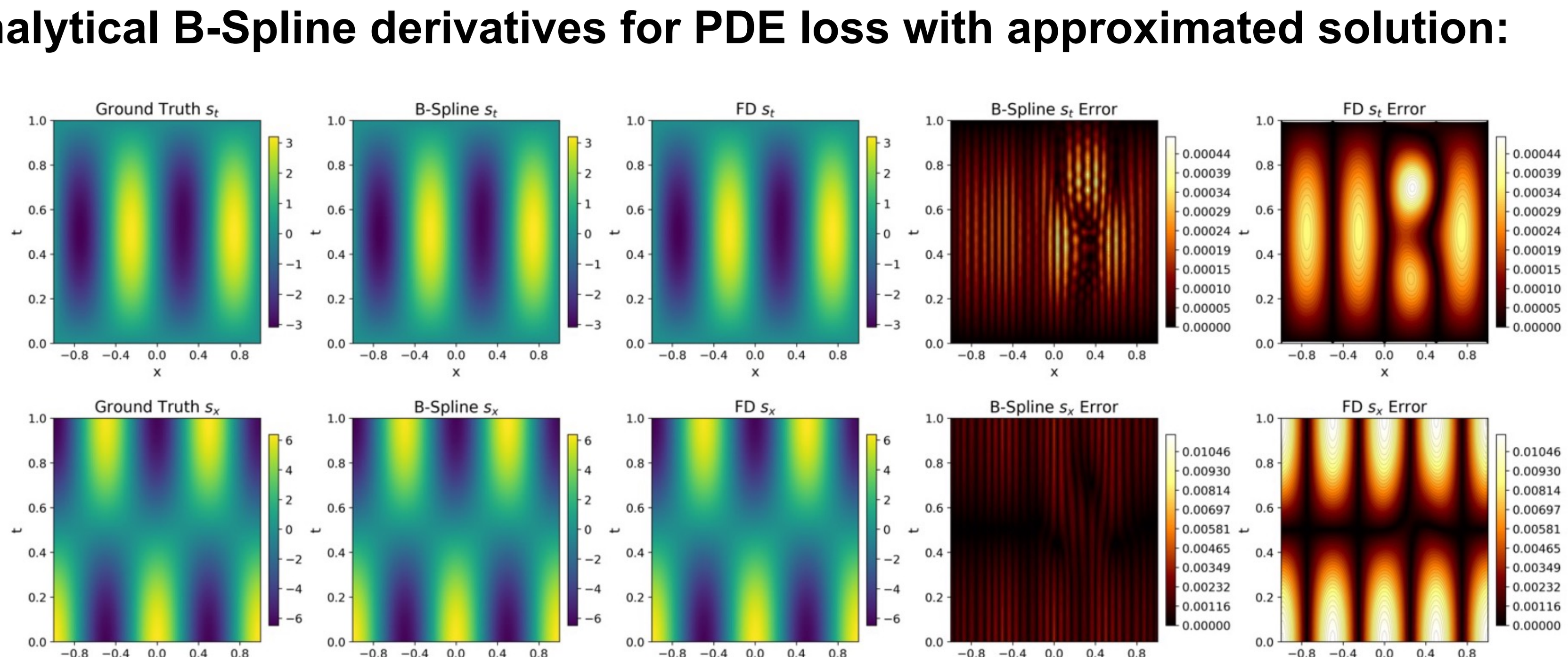
$$\hat{s}(x) = \sum_{i=1}^{\ell} c_i B_{i,d}(x)$$



Physics-Informed Deep B-Spline Networks (PI-BSNet):



Analytical B-Spline derivatives for PDE loss with approximated solution:



Theoretical Analysis

Definitions:

- PDE and ICBC parameters u and α in bounded domain $\mathcal{U}$ and $\mathcal{A}$
- Ground truth PDE solution $s_{u,\alpha}(x,t)$, where $(x,t) \in [a_1, b_1]\times\cdots\times[a_n, b_n] \subset \mathbb{R}^n$.
- PI-BSNet prediction $\hat{s}_{u,\alpha}(x,t)$

Theorem (universal approximator):

For any PDE and ICBC parameters $u \in \mathcal{U}$ and $\alpha \in \mathcal{A}$, with corresponding solution $s_{u,\alpha}$, there exists a PI-BSNet configuration such that for any $\epsilon > 0$,

$$\|\hat{s}_{u,\alpha} - s_{u,\alpha}\|_2 < \epsilon,$$

where $\|\cdot\|_2$ is the L_2 norm.

Theorem (generalization error bound):

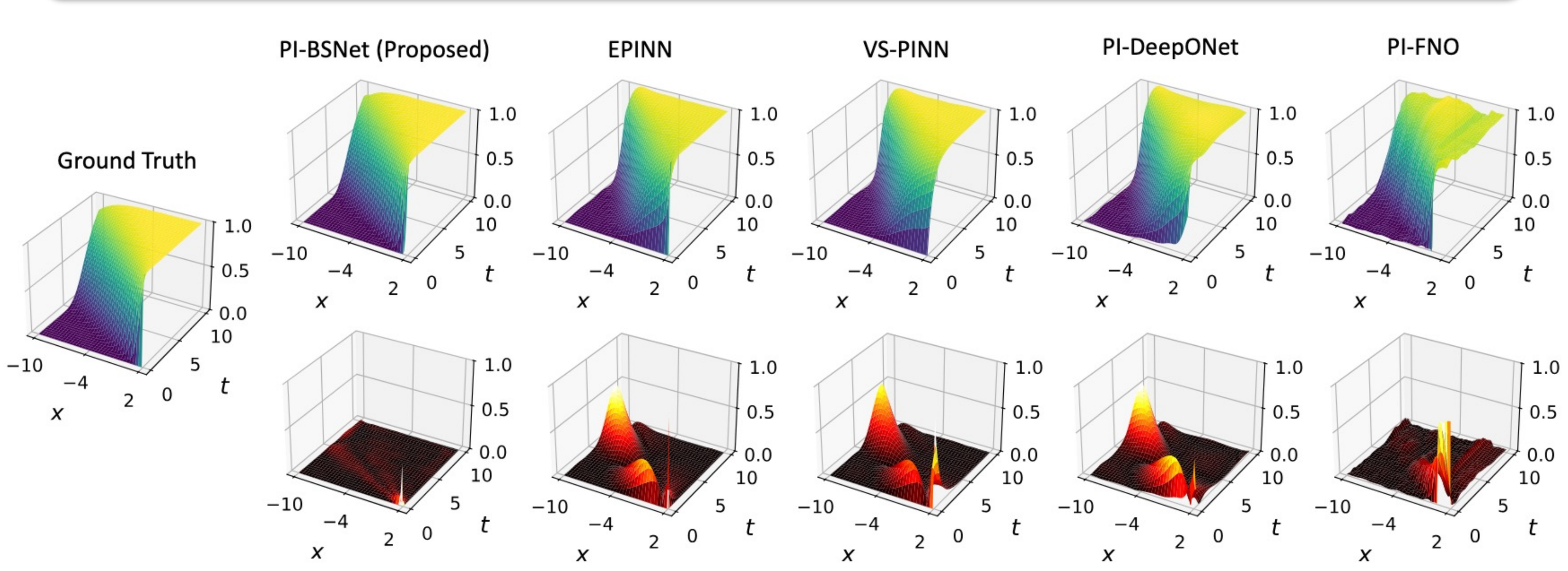
For any elliptic or parabolic PDE, for any $u \in \mathcal{U}$ and $\alpha \in \mathcal{A}$, given the taining loss enforcement density Δu and $\Delta\alpha$ and final training loss δ, the prediction error is bounded by

$$\sup_{(x,t)} |\hat{s}_{u,\alpha}(x,t) - s_{u,\alpha}(x,t)| < c\delta + L(\Delta u + \Delta\alpha),$$

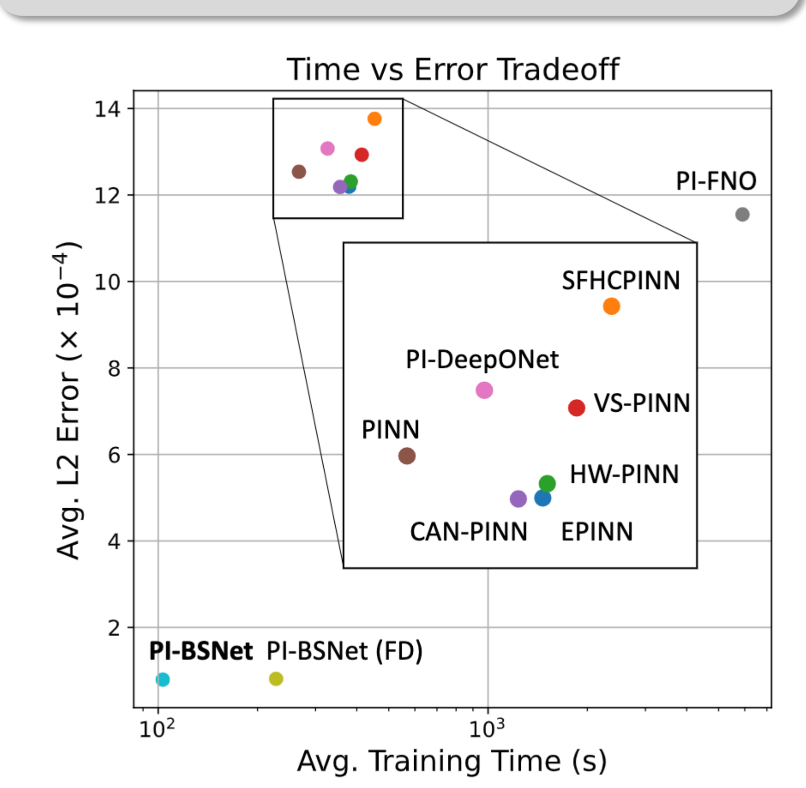
where c, L are constants.

Experiments

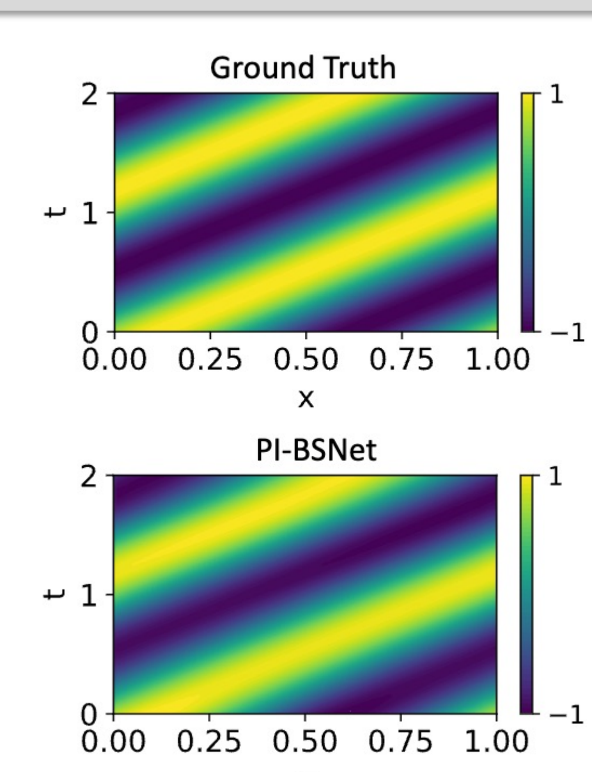
1. Assured boundary compliance



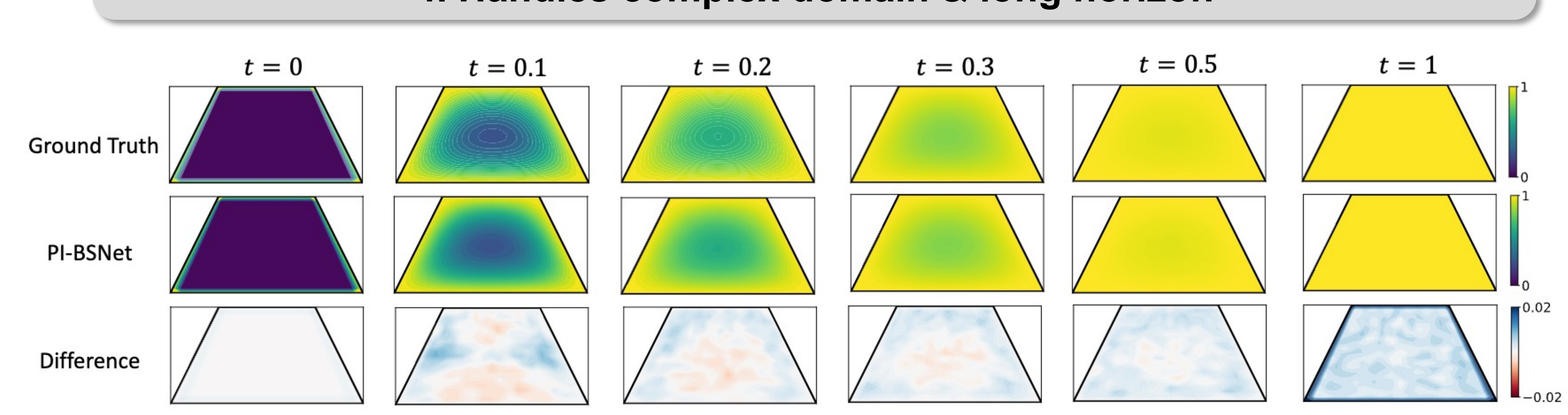
2. Efficient and accurate learning



3. Handles nonhomogeneous ICBC



4. Handles complex domain & long horizon



References

[1] Wang, Zhuoyuan, Raffaele Romagnoli, Saviz Mowlavi, and Yorie Nakahira. "Physics-Informed Deep B-Spline Networks." *Transactions on Machine Learning Research* (2026).

[2] Wang, Zhuoyuan, Raffaele Romagnoli, Kamyar Azizzadenesheli, and Yorie Nakahira. "Neural Spline Operators for Risk Quantification in Stochastic Systems." *In 2025 IEEE 64th Conference on Decision and Control (CDC)*, 2025.